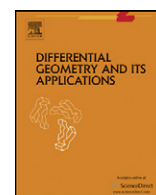




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Singularities of flat extensions from generic surfaces with boundaries

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ABSTRACT

We solve the problem on flat extensions of a generic surface with boundary in Euclidean 3-space, relating it to the singularity theory of the envelope generated by the boundary. We give related results on Legendre surfaces with boundaries via projective duality and observe the duality on boundary singularities. Moreover we give formulae related to remote singularities of the boundary-envelope.

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1. Introduction

We mean by *the flat extension problem* the problem on the existence, uniqueness and singularities of extensions of a surface across its boundary by flat surfaces in Euclidean 3-space \mathbf{R}^3 :

Problem. Let (S, γ) be a C^∞ surface with boundary γ in \mathbf{R}^3 . Find a C^1 -extension \tilde{S} of S such that $\tilde{S} \setminus \text{Int } S$ is C^∞ and the Gaussian curvature

$$K|_{\tilde{S} \setminus \text{Int } S} \equiv 0.$$

We call \tilde{S} a flat C^1 -extension of S . Then the surface (S, γ) with boundary is extended by a flat surface $(S', \gamma) = (\tilde{S} \setminus \text{Int } S, \gamma)$ with boundary. Recall that a surface S' in \mathbf{R}^3 is called *flat* if it is locally isometric to the plane and the condition is equivalent to that $K|_{S'} = 0$ [27]. Note that, in general, for a hypersurface $y = f(x_1, \dots, x_n)$ in \mathbf{R}^{n+1} , the Gauss–Kronecker curvature is given by

$$K = \frac{(-1)^n \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)}{\left[1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \dots + \left(\frac{\partial f}{\partial x_n}\right)^2\right]^{\frac{n+2}{n}}}.$$

Therefore, for a C^2 -extension \tilde{S} , K must be continuous on \tilde{S} . Thus, if S is not flat in itself, then we have to impose just C^1 -condition to the flat extensions \tilde{S} .

The efforts to solve the problem leads us to an insight on elementary differential geometry from singularity theory. We succeed the basic methods of geometric singularity theory [6,30]. In fact we assume that the surface with boundary (S, γ) is generic in this Introduction. However some of results hold for a surface *of finite type*: Regard it as a surface in the projective 3-space \mathbf{RP}^3 and take its projective dual S^\vee with boundary $\hat{\gamma}$ in the dual space \mathbf{RP}^{3*} . Then the condition is that both γ and $\hat{\gamma}$ are *of finite type* in the sense explained in Section 2. Under the condition the tangent lines and the osculating planes to γ (resp. $\hat{\gamma}$) are well-defined. Generic surfaces with boundary are of finite type.

A point $p \in \gamma$ is called an *osculating-tangent point* if the tangent plane $T_p S$ coincides with the osculating plane of γ , regarded as a space curve, at p .

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Theorem 1.1. (The solution to generic flat extension problem.) Let (S, γ) be a generic C^∞ surface with boundary γ in \mathbf{R}^3 . Then (S, γ) has a unique C^1 flat extension \tilde{S} locally across γ near $p \in \gamma$ provided p is not an osculating-tangent point for (S, γ) .

Remark 1.2. Let $g: S \rightarrow S^2$ be the Gauss mapping on S in \mathbf{R}^3 [3]. Then the local uniqueness of flat extensions holds under the weaker condition that the spherical curve $g|_\gamma: \gamma \rightarrow S^2$ is immersive.

In fact, to obtain the flat extension of (S, γ) along the boundary γ , we take tangent planes to S along γ and take the envelope of the one-parameter family of tangent planes. (See Section 2. See also [28].) We call it the *boundary-envelope* of (S, γ) . Then we have

Theorem 1.3. For a generic C^∞ surface (S, γ) with boundary, the singularities of boundary-envelope of (S, γ) are just cuspidal edges and swallowtails.

Remark 1.4. The folded umbrella (or the cuspidal cross-cap) [7] does not appear as a generic singularity of boundary-envelope. It appears in a generic one parameter family of boundary-envelope (cf. Lemma 2.17(2)).

Example 1.5. Let S be a C^∞ surface in \mathbf{R}^3 parametrised as

$$(x_1, x_2, x_3) = (t^2 + u, t, t^3 + ut)$$

with the parameters t and u , the boundary γ being given by $\{u = 0\}$, namely, by $\gamma(t) = (t^2, t, t^3)$. The osculating plane to γ at $t = 0$ is given by $\{x_3 = 0\}$ which is equal to the tangent plane of S at the origin. Thus the origin is a osculating-tangent point of (S, γ) . Then the boundary-envelope of (S, γ) is given by

$$(x, t) \mapsto (x_1, x_2, x_3) = (3t^2 - 2xt, x, -2t^3 + xt^2).$$

Its singular locus passes through the origin.

Example 1.6. Let S be a C^∞ surface in \mathbf{R}^3 parametrised as

$$(x_1, x_2, x_3) = (t + 1, 4t^3 - 2t^2 - 2t + u, 3t^4 - t^3 - t^2 + ut)$$

with the parameters t and u , the boundary γ being given by $\{u = 0\}$, namely, by

$$\gamma(t) = (t + 1, 4t^3 - 2t^2 - 2t, 3t^4 - t^3 - t^2).$$

Then the boundary-envelope of (S, γ) is given by

$$(x, t) \mapsto (x_1, x_2, x_3) = (x, 4t^3 - 2xt, 3t^4 - xt^2).$$

In general a map-germ $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ is called a *swallowtail* (or of type A_3) if it is diffeomorphic, i.e. C^∞ right-left equivalent, to the germ $(x, t) \mapsto (x, 4t^3 - 2xt, 3t^4 - xt^2)$ at $(0, 0)$. Moreover a map-germ $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$ is called a *cuspidal edge* (or of type A_2) if it is C^∞ right-left equivalent to the germ $(x, t) \mapsto (x, 3t^2 - 2xt, 2t^3 - xt^2)$ at $(0, 0)$.

In our example, the cuspidal edge and the swallowtail singularities are realised by a flat surface, a C^∞ surface which is flat outside the singular locus.

Note that, in the above example, the dual surface S^\vee is given by

$$(y_0, y_1, y_2) = (t^4 + u(t + 1), t^2 + u, t)$$

and its boundary $\hat{\gamma}$ is given in (y_0, y_1, y_2) -space, by

$$\hat{\gamma}(t) = (y_0(t), y_1(t), y_2(t)) = (t^4, t^2, t),$$

while $\hat{\gamma}^*$ is given by

$$\hat{\gamma}^*(t) = (x_1, x_2, x_3) = (6t^2, -8t^3, -3t^4),$$

for the notations which will be introduced in Section 2. The singular locus of the boundary-envelope of (S, γ) is given by $\hat{\gamma}^*$.

Motivated by this geometric method, we distinguish several “landmarks”, added to osculating-tangent points, on the boundary γ for a generic surface (S, γ) : A *parabolic point* on the boundary γ is a point on the intersection of the parabolic locus of S , the singular locus of the Gauss mapping $g: S \rightarrow S^2$ and γ [3]. A point $p \in \gamma$ is called a *swallowtail-tangent point* if the tangent plane $T_p S$ contacts with the envelope at a swallowtail point of the envelope. It turns out that a point $t = t_1$ of the parametric boundary γ is a swallowtail-tangent point if and only if, at $t = t_1$, the dual curve $(\hat{\gamma})^*$ to the dual-boundary $\hat{\gamma}$ is defined and the tangent line to the point $(\hat{\gamma})^*$ at $t = t_1$ contains the swallowtail point of the envelope $(\hat{\gamma})^\vee$.

Parabolic points on γ for (S, γ) correspond to singular points of the dual S^\vee on the dual-boundary $\widehat{\gamma}$.

In [Example 1.6](#), γ has no osculating-tangent point nor parabolic point, but it has one swallowtail-tangent point at $(0, 0, 0)$.

By [Theorem 1.1](#), a generic surface with boundary (S, γ) has a local flat extension across non-osculating-tangent points. In fact, at any osculating-tangent point, the singular locus of the boundary-envelope passes through the boundary at that point. See [Section 2](#) for the exact classification of singularities of the local extension problem. Moreover a global obstruction occurs by singularities of the envelope, in particular, by self-intersection loci. Thus a swallowtail point of the envelope provides “a global obstruction with local origin” for the flat extension problem. With this motivation, we characterise the osculating tangent points and the swallowtail-tangent points in terms of Euclidean invariant of the surface-boundary γ of S .

To characterise these landmarks, we recall three fundamental invariants κ_1, κ_2 and κ_3 of the boundary γ in [Section 3](#). Actually κ_1 is the *geodesic curvature*, κ_2 is the *normal curvature* and κ_3 is the *geodesic torsion* of γ , up to sign. These three invariants are defined for any immersed space curve with a framing.

Remark 1.7. The curvature κ and the torsion τ of γ as a space curve is related to κ_1, κ_2 and κ_3 by

$$\kappa = \sqrt{\kappa_1^2 + \kappa_2^2},$$

$$\tau = \kappa_3 + \left(\frac{\kappa}{\kappa_1}\right)\left(\frac{\kappa_2}{\kappa}\right)' = \kappa_3 - \left(\frac{\kappa}{\kappa_2}\right)\left(\frac{\kappa_1}{\kappa}\right)' = \kappa_3 + \frac{\kappa_1\kappa_2' - \kappa_2\kappa_1'}{\kappa_1^2 + \kappa_2^2},$$

for the arc-length differential, provided $\kappa_1 \neq 0$ and $\kappa_2 \neq 0$. Note that the torsion τ of an immersed space curve is defined when the curvature $\kappa \neq 0$. Moreover it can be shown that, for any space curve γ with curvature κ and τ ($\kappa \neq 0$), and given any three functions κ_1, κ_2 and κ_3 on the curve satisfying the above relations, there exists a surface S with boundary γ such that the three invariants coincide with the given κ_1, κ_2 and κ_3 .

Then our generic characterisation is given by

Theorem 1.8. Let (S, γ) be a generic C^∞ surface with boundary in Euclidean three space \mathbf{R}^3 . Then the osculating-tangent point on γ is characterised by the condition $\kappa_2 = 0$.

Moreover, we show that there exists a characterisation of the swallowtail-tangent points in terms of $\kappa_1, \kappa_2, \kappa_3$ and their derivatives of order ≤ 3 . In fact we have

Theorem 1.9 (Euclidean generic characterisation of swallowtail-tangent). Let (S, γ) be a generic C^∞ surface with boundary in Euclidean three space \mathbf{R}^3 . A swallowtail-tangent point of γ is characterised by the condition

$$\begin{aligned} \text{(I)} \quad & \kappa_2 \neq 0, \\ \text{(II)} \quad & \kappa_1^2\kappa_3(\kappa_2^2 + \kappa_3^2) + \kappa_2(\kappa_2^2 + \kappa_3^2)\kappa_1' - 3\kappa_1\kappa_3^2\kappa_2' + 3\kappa_1\kappa_2\kappa_3\kappa_3' + 2\kappa_3(\kappa_2')^2 - 2\kappa_2\kappa_2'\kappa_3' - \kappa_2\kappa_3\kappa_2'' + \kappa_2^2\kappa_3'' = 0, \\ \text{(III)} \quad & 2\kappa_1\kappa_3^3(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) + 2\kappa_1\kappa_3(2\kappa_2^2 + \kappa_3^2)\kappa_1' + (3\kappa_2^2 - 2\kappa_3^2)\kappa_1'\kappa_2' + 5\kappa_2\kappa_3\kappa_1'\kappa_3' + 3\kappa_1\kappa_2(\kappa_3')^2 + \kappa_2(3\kappa_1\kappa_2 + \kappa_2^2 + \kappa_3^2)\kappa_1'' \\ & + 3\{\kappa_1(-\kappa_2' - \kappa_3^2 + \kappa_2\kappa_3) + 3(\kappa_3\kappa_2' - \kappa_2\kappa_3')\}\kappa_3'' + \kappa_2(\kappa_2 - 2\kappa_3)\kappa_3''' \neq 0. \end{aligned}$$

Remark 1.10. The existence of an osculating-tangent point on the boundary γ depends on the geometry of the surface S itself.

For example, on an elliptic surface, there does not exist any osculating-tangent point. The surface is necessarily hyperbolic near an osculating-tangent point with $\kappa_2 = 0, \kappa_3 \neq 0$.

We are interested in the interaction between singularity and geometry. In our topic of this paper, local geometry of surface-curve provides a global effect to the singularity of the envelope. In fact we give the exact formula for the distance between the swallowtail-tangent point on the surface-boundary and the swallowtail point on the boundary-envelope (envelope-swallowtail) in terms of local geometric invariants of the boundary. (See also [Proposition 3.6](#).)

Proposition 1.11. The distance d between the swallowtail-tangent point on the surface-boundary and the envelope-swallowtail is given by

$$d = \left| \frac{\kappa_2 \sqrt{\kappa_2^2 + \kappa_3^2}}{\kappa_2(\kappa_3' + \kappa_1\kappa_2) + \kappa_3(-\kappa_2' + \kappa_1\kappa_3)} \right|.$$

Remark 1.12. If the denominator of the above formula vanishes, then the formula reads $d = \infty$, and, in fact, the envelope-swallowtail lies at infinity. If $\kappa_2 = 0$, then the formula reads $d = 0$, and, in fact, the non-generic coincidence of an osculating-tangent point and a swallowtail-tangent point occurs, and the envelope-swallowtail coincides with the swallowtail-tangent point.

In Section 2, we give the background for the basic results on projective duality of Legendre surfaces with boundaries (Theorems 2.2 and 2.15). As a corollary we show Theorem 1.1. In Section 3, we show the Euclidean characterisations of osculating-tangent points and swallowtail-tangent points (Theorems 1.8 and 1.9) and the distance formula (Proposition 1.11) in more general setting: Our perspective though singularity theory extends the results on generic surface-boundaries to more general surface-boundaries. Lastly the local flat extension problem is solved naturally as a by-product of other results in this paper.

A local geometry of surface-boundary causes a global effect to the singularities of boundary-envelope. Thus we provide examples of results on the interaction between singularity and geometry and between local and global.

Apart from the flat extension problem, also there exist several extension problems: For instance we can consider the C^1 extension problem by a surface with $K = c$ for a non-zero constant c . Note that generically a surface of constant Gaussian curvature has only cuspidal edges and swallowtails as singularities [15].

Some of the results in this paper have been announced in the monograph [14].

2. Projective geometry on singularities of front-boundaries

To study the existence, uniqueness and singularities of flat extensions by the geometric method, we recall several basic results on Legendre surfaces with boundaries in projective-contact framework [4].

The projective duality between the projective $(n+1)$ -space $\mathbf{RP}^{n+1} = P(\mathbf{R}^{n+2})$ and the dual projective $(n+1)$ -space $\mathbf{RP}^{n+1*} = P(\mathbf{R}^{n+2*})$ is given through the incidence manifold

$$I^{2n+1} = \{([X], [Y]) \in \mathbf{RP}^{n+1} \times \mathbf{RP}^{n+1*} \mid X \cdot Y = 0\},$$

and projections $\pi_1: I^{2n+1} \rightarrow \mathbf{RP}^{n+1}$ and $\pi_2: I^{2n+1} \rightarrow \mathbf{RP}^{n+1*}$. The space I is identified with the space $PT^*\mathbf{RP}^{n+1}$ of contact elements of \mathbf{RP}^{n+1} and with $PT^*\mathbf{RP}^{n+1*}$ as well. See [25] for instance. It is endowed with the natural contact structure

$$D = \{X \cdot dY = 0\} = \{dX \cdot Y = 0\} \subset TI \cong T(PT^*\mathbf{RP}^{n+1}) \cong T(PT^*\mathbf{RP}^{n+1*}).$$

A C^∞ hypersurface S in \mathbf{RP}^{n+1} lifts uniquely to the Legendre hypersurface L in I which is an integral submanifold to D :

$$L = \{([X], [Y]) \in I \mid [X] \in S, [Y] \text{ determines } T_{[X]}S \text{ as a projective hyperplane}\}.$$

Then L projects to \mathbf{RP}^{n+1*} by π_2 . The “front” $S^\vee = \pi_2(L)$, as a parametrised hypersurface with singularities, is called the *projective dual* or *Legendre transform* of S [2].

If we start with a surface S with boundary γ in \mathbf{RP}^3 , $n = 2$, then the Legendre lift L also has the boundary Γ :

$$\Gamma = \{([X], [Y]) \in L \mid [X] \in \gamma\} = \partial L.$$

Then L is a Legendre surface and Γ is an integral curve in I^5 to the contact distribution D :

$$T\Gamma \subset TL \subset D \subset TI.$$

Now we have a Legendre surface with boundary (L, Γ) in I and two Legendre fibrations π_1, π_2 :

$$\begin{array}{ccccc} (L, \Gamma) & \subset & PT^*\mathbf{RP}^3 & \cong & I^5 & \cong & PT^*\mathbf{RP}^{3*} \\ \downarrow & & \downarrow & \swarrow \pi_1 & & \searrow \pi_2 & \downarrow \\ (S, \gamma) & \subset & \mathbf{RP}^3 & & & & \mathbf{RP}^{3*} \end{array}$$

We identify Γ with the inclusion map $\Gamma \hookrightarrow I$. Then we get the triple of Legendre surfaces (L, L_1, L_2) possibly with singularities in I :

$$L_1 = \{([X], [Y]) \mid [X] \in \pi_1(\Gamma), [Y] \text{ is a tangent plane to } \pi_1 \circ \Gamma \text{ at } [X]\},$$

the projective conormal bundle of the space curve $\pi_1(\Gamma)$, and

$$L_2 = \{([X], [Y]) \mid [Y] \in \pi_2(\Gamma), [X] \text{ is a tangent plane to } \pi_2 \circ \Gamma \text{ at } [Y]\}$$

the projective conormal bundle of the space curve $\pi_2(\Gamma)$.

Moreover, the *dual surface* of the space curve $\pi_1(\Gamma)$ (resp. $\pi_2(\Gamma)$) is defined as the front $\pi_2(L_1)$ (resp. $\pi_1(L_2)$). Thus we have two fronts or frontal surfaces $\pi_1(L), \pi_1(L_2) \subset \mathbf{RP}^3$ and $\pi_2(L), \pi_2(L_1) \subset \mathbf{RP}^{3*}$ respectively.

Starting from C^∞ surface (S, γ) with boundary in \mathbf{RP}^3 , we have the Legendre-integral lifting (L, Γ) in I^5 . Then the *boundary-envelope* of (S, γ) is defined by $\pi_1|_{L_2}: L_2 \rightarrow \mathbf{RP}^3$. Moreover $\pi_2|_{L_1}$ gives the boundary-envelope of the dual $(S^\vee, \hat{\gamma})$.

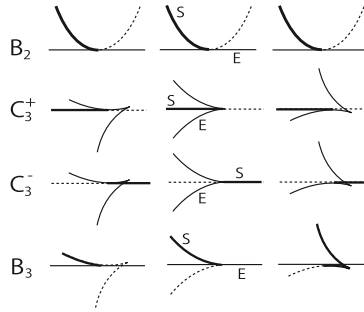
Remark 2.1. In the above definition of “projective conormal bundle” L_2 , the interpretation of “tangent plane” is not unique if $\pi_2 \circ \Gamma$ is not an immersion. In this paper we mainly concern with the generic case where $\pi_2 \circ \Gamma$ is an immersion (cf. Theorem 2.2(3). See also Remark 2.11).

We call a pair of germs of fronts (S, E) , say $(\pi_1|_L, \pi_1|_{L_1})$, is of type B_2 (resp. B_3, C_3) if it is diffeomorphic, i.e. C^∞ right-left equivalent, to the following local model as a multi-germ:

$$B_2 : (t_1, t_2) \mapsto (t_1, t_2, t_2^2), \quad \pm t_2 \geq 0, \quad (s_1, s_2) \mapsto (s_1, s_2, 0),$$

$$C_3^\pm : (t_1, t_2) \mapsto (t_1, t_2, 0), \quad \pm t_2 \geq 0, \quad (s_1, s_2) \mapsto (s_1, -3s_2^2 - 2s_1s_2, 2s_2^3 + s_1s_2^2),$$

$$B_3 : (t_1, t_2) \mapsto (t_1, -3t_2^2 - 2t_1t_2, 2t_2^3 + t_1t_2^2), \quad \pm t_2 \geq 0, \quad (s_1, s_2) \mapsto (s_1, s_2, 0) \quad (\pm \text{ for } B_2, B_3 \text{ give the same class}).$$



Then we have the basic results:

Theorem 2.2. For a generic Legendre surface with boundary (L, Γ) in the incident manifold $I^5 \cong PT^*\mathbf{RP}^3 \cong PT^*\mathbf{RP}^{3*}$ with respect to C^∞ topology, we have

- (1) The singularities of $\pi_1|_L$ and $\pi_2|_L$ are just cuspidal edges and swallowtails.
- (2) The diffeomorphism types of the pair $(\pi_1|_L, \pi_1|_{L_2})$ (resp. $(\pi_2|_L, \pi_2|_{L_1})$) of germs at points on Γ are given by B_2, B_3 and C_3 .
- (3) Both $\pi_1|_\Gamma$ and $\pi_2|_\Gamma$ are generically immersed space curves in the sense of Scherbak ("Scherbak-generic") [25], in \mathbf{RP}^3 and \mathbf{RP}^{3*} respectively. Singularities of $\pi_1|_{L_2}$ and $\pi_2|_{L_1}$ are only cuspidal edges and swallowtails.

Remark 2.3. We can show that the singular loci of $\pi_1|_L$ and $\pi_2|_L$, and Γ are in general position in L and moreover that the swallowtail points of $\pi_1|_L$ and $\pi_2|_L$ are not on the intersections of the above three curves.

For the point (1), it is well known that the stable front $\pi_1(L)$ has A_ℓ -singularities ($\ell \leq 3$) by Legendre singularity theory [2]. The cuspidal edge singularity is called of type A_2 and the swallowtail singularity is called of type A_3 , while A_1 means regular. For the point (2), it is well known that the stable front with boundary $(\pi_1(L), \pi_1(L_2))$ has B_ℓ or C_ℓ -singularity ($\ell \leq 3$) by the theory of boundary singularities; we know the diffeomorphism types of stable fronts with boundary [1,2]. See also [23,24,8,29]. Moreover, for the point (2), we remark that, the duality of boundary singularities found by I.G. Scherbak, the "Scherbak duality" [23,24] are realised via Legendre duality in our geometrical situation: The C_3 -singularity appears at an osculating-tangent point on γ in \mathbf{RP}^3 and B_3 -singularity appears at a point in \mathbf{RP}^{3*} corresponding to a parabolic point on γ .

These basic results are proved by the standard methods in singularity theory: Here we use Legendre-integral version of relative transversality theorem [10,13] to make assure ourselves. A Legendre immersion $i : (L, \Gamma) \rightarrow I^5$ is approximated by $i' : (L, \Gamma) \rightarrow I^5$ such that the r -jet extension $j^r i'$ is transverse to given a finite family of submanifolds in the isotropic jet space $J_{\text{int}}^r(L, I)$ and $(j^r i')|_\Gamma : \Gamma \rightarrow J_{\text{int}}^r(L, \Gamma; I, I)$ is transverse to given a finite family of submanifolds in the relative isotropic jet space $J_{\text{int}}^r(L, \Gamma; I, I)$ which is a fibration over $\Gamma \times I$ [10]. Moreover $j^r(i'|_\Gamma)$ is transverse to given finite family of submanifolds in $J_{\text{int}}^r(\Gamma, I)$. We will give a proof of the Legendre (or integral) transversality theorem, because it seems to be never explicitly given.

Theorem 2.4. (Integral transversality theorem [9,13].) Let (I^{2n+1}, D) be a $(2n+1)$ -dimensional contact manifold, M^m an m -dimensional manifold ($m \leq n$) and $f : M \rightarrow I$ an integral immersion to the contact structure $D \subset TI$. Let $r \in \mathbf{N}$ and Q_λ ($\lambda \in \Lambda$) a finite family of submanifolds of $J_{\text{int}}^r(M, I)$. Then f is approximated, in the Whitney C^∞ topology, by an integral immersion $f' : M \rightarrow I$ such that the r -jet extension $j^r f' : M \rightarrow J_{\text{int}}^r(M, I)$ is transverse to all Q_λ ($\lambda \in \Lambda$).

Proof. First recall that the space of integral immersion-jets $J_{\text{int}}^r(M, I)$ is a submanifold of $J^r(M, I)$ [9]. Then we follow the standard construction of [18] in the integral context: Suppose, near each point $p \in M$ and $f(p) \in I$, f is represented as

$$(t_1, \dots, t_m) \mapsto (t_1, \dots, t_m, 0, \dots, 0)$$

by a local coordinates of (M, p) and a local Darboux coordinates of $(I, f(p))$. Denote by $P(m, \ell; k)$ the space of polynomial mappings $\mathbf{R}^m \rightarrow \mathbf{R}^\ell$ of degree $\leq k$. Let E be a neighbourhood of $(0, \text{id}_{\mathbf{R}^m})$ of $P(m, 1; r+1) \times P(m, m; r)$. Choose a C^∞ function $\rho: \mathbf{R}^m \rightarrow [0, 1]$ with a compact support. For (S, σ) , set

$$\varphi(S, \sigma)(t) = \left(t, \rho(t) \frac{\partial S}{\partial t} \right) \circ \sigma(t),$$

and extend it to an integral immersion $\varphi(S, \sigma): M \rightarrow I$. Then $\Phi: E \times I \rightarrow J_{\text{int}}^r(M, I)$ defined by $\Phi(S, \sigma, t) = j^r(\varphi(S, \sigma))(t)$ is a submersion at $(0, \text{id}, p)$ and transverse to Q_λ locally. Then the result follows by Sard's theorem. \square

Remark 2.5. The relative version of Theorem 2.4 is also valid, similarly to the construction of [10]: Let $N^\ell \subset M^m$ be a submanifold and $r \in \mathbf{N}$. Then we consider the relative integral jet space $J_{\text{int}}^r(M, N, I, I)$ fibered over $N \times I$ with the fibre $J_{\text{int}}^r(m, 2n+1)$, the space of jets of integral immersion-germs $(\mathbf{R}^m, 0) \rightarrow (\mathbf{R}^{2n+1}, 0)$ to a local model \mathbf{R}^{2n+1} of the contact space. Let Q_λ ($\lambda \in \Lambda$) be a finite family of submanifolds of $J_{\text{int}}^r(M, I)$, $R_{\lambda'}$ ($\lambda' \in \Lambda'$) a countable family of submanifolds of $J_{\text{int}}^r(M, N, I, I)$ and $P_{\lambda''}$ ($\lambda'' \in \Lambda''$) a countable family of submanifolds of $J_{\text{int}}^r(N, I)$. Then any integral immersion $f: M \rightarrow I$. Let $r \in \mathbf{N}$ is approximated, in the Whitney C^∞ topology, by an integral immersion $f': M \rightarrow I$ such that the r -jet extension $j^r f': M \rightarrow J_{\text{int}}^r(M, I)$ is transverse to all Q_λ ($\lambda \in \Lambda$), $j^r f'|_N: N \rightarrow J_{\text{int}}^r(M, N, I, I)$ is transverse to all $R_{\lambda'}$ ($\lambda' \in \Lambda'$) and $j^r(f'|_N): N \rightarrow J_{\text{int}}^r(N, I)$ is transverse to all $P_{\lambda''}$ ($\lambda'' \in \Lambda''$).

The genericity for the points (1), (2) is described in terms of *generating families*: In the affine open subset $U \times V = \{X_0 \neq 0, Y_3 \neq 0\}$ of $\mathbf{R}P^3 \times \mathbf{R}P^{3*}$, we set $x_i = -X_i/X_0$, $y_j = -Y_{3-j}/Y_3$ ($1 \leq i, j \leq 3$), and

$$F(x_1, x_2, x_3, y_1, y_2, y_3) = -y_3 + x_1 y_2 + x_2 y_1 - x_3.$$

Then $I \cap (U \times V)$ is defined by $F = 0$.

Let $L = \{(x_1(u, v), x_2(u, v), x_3(u, v), y_1(u, v), y_2(u, v), y_3(u, v))\}$ be a Legendre surface in $I \cap (U \times V)$ parametrised by $(u, v) \in \mathbf{R}^2$. Then we have two families of functions $F_2, F_1: \mathbf{R}^2 \times \mathbf{R}^3 \rightarrow \mathbf{R}$,

$$F_2(u, v; y_1, y_2, y_3) = F(x_1(u, v), x_2(u, v), x_3(u, v), y_1, y_2, y_3),$$

$$F_1(u, v; x_1, x_2, x_3) = F(x_1, x_2, x_3, y_1(u, v), y_2(u, v), y_3(u, v)).$$

Then F_1 (resp. F_2) is a generating family for $\pi_1|_L$ (resp. $\pi_2|_L$). Similarly, for an integral curve $\Gamma = \{(x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t))\}$, we set

$$G_2(t; y_1, y_2, y_3) = F(x_1(t), x_2(t), x_3(t), y_1, y_2, y_3),$$

$$G_1(t; x_1, x_2, x_3) = F(x_1, x_2, x_3, y_1(t), y_2(t), y_3(t)).$$

Then G_1 (resp. G_2) is a generating family for $\pi_1|_{L_2}$ (resp. $\pi_2|_{L_1}$). Note that $G_i = F_i|_{\gamma \times \mathbf{R}^3}$, $i = 1, 2$. The local singularities of pair of fronts $(\pi_1|_L, \pi_1|_{L_2})$ (resp. $(\pi_2|_L, \pi_2|_{L_1})$) are represented, via the analysis of generating families, as strata in the integral jet spaces. Let us make clear the relation of transversality in Legendre jet space and that in the jet space of generation functions: We use the following basic method to show genericity:

Proposition 2.6. Let I^{2n+1} be a $(2n+1)$ -dimensional contact manifold, $\pi: I \rightarrow B^{n+1}$ a Legendre fibration, and N an n -manifold. Let $f: N \rightarrow I$ be a Legendre immersion, $u_0 \in N$ and $F: (N \times \Lambda, (u_0, \lambda_0)) \rightarrow \mathbf{R}$ a generating family of $f: (N, u_0) \rightarrow I$. Then we have

- (1) $f: (N, u_0) \rightarrow I$ is Legendre stable if and only if F is \mathcal{K} -stable unfolding of $F|_{N \times \{\lambda_0\}}$.
- (2) $f: (N, u_0) \rightarrow I$ is Legendre stable if and only if $j^{n+1} f: (N, u_0) \rightarrow J_{\text{int}}^{n+1}(N, I)$ is transversal to the Legendre orbit of $j^{n+1} f(u_0)$.
- (3) F is \mathcal{K} -stable unfolding of $F|_{N \times \{\lambda_0\}}$ if and only if $j_1^{n+2} F: (\Lambda, \lambda_0) \rightarrow J^{n+2}(N, \mathbf{R})$, defined by $j_1^{n+2} F(\lambda) = j^{n+2}(F|_{N \times \{\lambda\}})(u_0)$ is transverse to \mathcal{K} -orbit of $j^{n+2}(F|_{N \times \{\lambda_0\}})(u_0)$.

Remark 2.7. It is known that any germ of Legendre stable Legendre immersion $f: (N^n, u_0) \rightarrow I^{2n+1}$ is $(n+1)$ -determined among Legendre immersion-germs [13]. Moreover, for its generating family $F: (N \times \Lambda, (u_0, \lambda_0)) \rightarrow \mathbf{R}$, $F|_{N \times \{\lambda_0\}}: (N \times \{\lambda_0\}, (u_0, \lambda_0)) \rightarrow \mathbf{R}$ is $(n+2)$ -determined. Note that the $(k+1)$ -jet of $F|_{N \times \{\lambda_0\}}$ is determined by the k -jet of f from the integrality condition.

Also we use the relative version:

Proposition 2.8. Let I^{2n+1} be a $(2n+1)$ -dimensional contact manifold, $\pi: I \rightarrow B^{n+1}$ a Legendre fibration, and $(N, \partial N)$ an n -manifold with boundary. Let $f: (N, \partial N) \rightarrow I$ be a Legendre immersion, $u_0 \in \partial N$ and $F: (N \times \Lambda, (u_0, \lambda_0)) \rightarrow \mathbf{R}$ a generating family of $f: (N, u_0) \rightarrow M$. Then we have

- (1) $f : (N, \partial N, u_0) \rightarrow I$ is Legendre stable if and only if F is \mathcal{K}_b -stable unfolding of $F|_{N \times \{\lambda_0\}}$.
 (2) $f : (N, \partial N, u_0) \rightarrow I$ is Legendre stable if and only if $j^{n+1}f|_{\partial N} : (\partial N, u_0) \rightarrow J_{\text{int}}^{n+1}(N, \partial N; I, I)$ is transversal to Legendre orbit of $j^{n+1}f(u_0)$.
 (3) F is \mathcal{K}_b -stable unfolding of $F|_{N \times \{\lambda_0\}}$ if and only if $j_1^{n+2}F : (\Lambda, \lambda_0) \rightarrow J^{n+2}(N, \partial N; \mathbf{R}, \mathbf{R})$, defined by $j_1^{n+2}F(\lambda) = j^{n+2}(F|_{N \times \{\lambda\}})(u_0)$ is transverse to \mathcal{K} -orbit of $j^{n+2}(F|_{N \times \{\lambda_0\}})(u_0)$.

In the above proposition, \mathcal{K}_b -equivalence means boundary \mathcal{K} -equivalence. The points (1), (3) are basic results in (boundary) singularity theory [2]. The point (2) follows the infinitesimal characterisation of Lagrange stability. See [9].

For the point (3) of Theorem 2.2, we have to know more information on the projective geometry of boundaries, $\gamma = \pi_1(\Gamma)$ and $\hat{\gamma} = \pi_2(\Gamma)$. We write $\gamma = \pi_1(\Gamma)$ and $\hat{\gamma} = \pi_2(\Gamma)$, and call $\hat{\gamma}$ the dual-boundary to γ .

To show the point (3), we recall some projective geometry-singularity in three space: We use, to a space curve c in \mathbf{RP}^3 (resp. in \mathbf{RP}^{3*}), the notions of the dual curve c^* and the dual surface c^\vee in \mathbf{RP}^{3*} (resp. \mathbf{RP}^3). Note that the dual-boundary \hat{c} is different from the dual curve c^* to c and it is defined only when c is regarded as a surface-curve or a framed curve.

A C^∞ space curve $\gamma : \mathbf{R} \rightarrow \mathbf{RP}^3$ is called of finite type at $t = t_0 \in \mathbf{R}$, if for each system of affine coordinates in \mathbf{RP}^3 near $\gamma(t_0)$, the $3 \times \infty$ matrix

$$(\gamma'(t_0), \gamma''(t_0), \dots, \gamma^{(r)}(t_0), \dots)$$

is of rank 3. Introduce the $3 \times r$ matrix

$$A_r(t) = (\gamma'(t), \gamma''(t), \dots, \gamma^{(r)}(t)).$$

Then the type (a_1, a_2, a_3) of γ at $t = t_0$ is define by

$$\begin{aligned} a_1 &= \min\{r \mid \text{rank } A_r(t_0) = 1\}, & a_2 &= \min\{r \mid \text{rank } A_r(t_0) = 2\}, \\ a_3 &= \min\{r \mid \text{rank } A_r(t_0) = 3\}. \end{aligned}$$

Remark that a_1, a_2, a_3 are positive integers with $a_1 < a_2 < a_3$ and that, for some system of affine coordinates centred at $\gamma(t_0)$, γ is expressed as

$$\begin{cases} X_1(t) = (t - t_0)^{a_1} + o((t - t_0)^{a_1}), \\ X_2(t) = (t - t_0)^{a_2} + o((t - t_0)^{a_2}), \\ X_3(t) = (t - t_0)^{a_3} + o((t - t_0)^{a_3}). \end{cases}$$

A point of γ of type $(1, 2, 3)$ is called an ordinary point. Otherwise, it is called a special point of γ . Special points are isolated on \mathbf{R} for a space curve of finite type.

Lemma 2.9. (See O.P. Scherbak [25].) A generic space curve γ in \mathbf{RP}^3 is of type $(1, 2, 3)$ or $(1, 2, 4)$ at each point.

Proof. Consider the 3-jet space

$$J^3(\mathbf{R}, \mathbf{RP}^3) = \{j^3\gamma(t_0) \mid \gamma = (X_0(t), X_1(t), X_2(t), X_3(t)) : (\mathbf{R}, t_0) \rightarrow \mathbf{RP}^3\}$$

of curves in \mathbf{RP}^3 . Set

$$\Sigma = \{j^3\gamma(t_0) \mid \det(\gamma(t_0), \gamma'(t_0), \gamma''(t_0), \gamma'''(t_0)) = 0\}.$$

The conditions are independent of the choice of homogeneous coordinates of γ .

Then Σ is a fibration over $\mathbf{R} \times \mathbf{RP}^3$ whose fibre is an algebraic hypersurface in the jet space $J^3(1, 3)$. A map-germ $\gamma : (\mathbf{R}, t_0) \rightarrow \mathbf{RP}^3$ is of type $(1, 2, 3)$ (resp. $(1, 2, 4)$) if and only if $j^3\gamma(t_0) \notin \Sigma$ (resp. $j^3\gamma(t_0) \in \Sigma$ and $j^3\gamma : (\mathbf{R}, t_0) \rightarrow J^3(\mathbf{R}, \mathbf{RP}^3)$ is transverse to Σ). Therefore, by the transversality theorem, we have the result. \square

We call a curve Scherbak-generic if it is of finite type of type $(1, 2, 3)$ or $(1, 2, 4)$ at any point.

The osculating planes to a space curve γ of finite type form a dual curve γ^* of the curve γ in the dual space.

Lemma 2.10 (Duality theorem). (See Arnol'd and Scherbak [25].)

- (1) The dual curve γ^* to a curve-germ γ of finite type (a_1, a_2, a_3) is a curve-germ of finite type $(a_3 - a_2, a_3 - a_1, a_3)$.
 (2) The dual surface to a curve-germ γ of finite type is the tangent developable of the dual curve γ^* of γ .

The tangent developable of γ is a surface ruled by tangent lines to γ [7,19,20,25,26,11].

Remark 2.11. The notion of dual surface depends on the notion of tangency (Remark 2.1). For the curves of finite type, the notion of tangent line is well defined. Therefore if $\pi_1 \circ \Gamma$ and $\pi_2 \circ \Gamma$ are both of finite type, then both L_1, L_2 are well-defined, so are both $\pi_1|_{L_2}$ and $\pi_2|_{L_1}$. Thus the notion of boundary-envelope is well-defined. Also note that if we start from the generating family to define the boundary-envelope, we get the “extended” envelope: To each singular point of $\pi_2 \circ \Gamma$ the hyperplane in \mathbf{RP}^3 which corresponds to it is added to the original envelope $\pi_1(L_2)$.

Lemma 2.12. *If γ is of type $(1, 2, 3)$, then γ^* is of type $(1, 2, 3)$, and the dual surface is diffeomorphic to the cuspidal edge. If γ is of type $(1, 2, 4)$, then γ^* is of type $(2, 3, 4)$, and the dual surface is diffeomorphic to the swallowtail.*

For the proof, consult the survey paper [11] on the singularities of tangent developables. We also remark

Lemma 2.13. *The dual surface of a space curve-germ γ of finite type is diffeomorphic to the cuspidal edge (resp. the swallowtail) if and only if the type of γ is equal to $(1, 2, 3)$ (resp. $(1, 2, 4)$).*

Note that the type of $\hat{\gamma}^*$ is $(1, 2, 3)$ (resp. $(2, 3, 4)$) if and only if $\hat{\gamma}$ is of type $(1, 2, 3)$ (resp. $(1, 2, 4)$). Then Lemma 2.13 follows from the following general result which does not stated in [11]:

Proposition 2.14. *Let γ, γ' be space curve-germs of finite types. If their tangent developables are diffeomorphic, then their types coincide.*

Proof. Let $\text{type}(\gamma) = (m, m + s, m + s + r)$. Then diffeomorphism-class of the tangent developable of γ is given by $\text{dev}(\gamma): (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^3, 0)$

$$x_1 = x, \quad x_2 = t^{s+m} + \cdots + x(t^s + \cdots), \quad x_3 = t^{r+s+m} + \cdots + x(ct^{r+s} + \cdots),$$

where (x, t) is a system of parameters, \cdots means higher order terms in t , and c is a non-zero constant [11,12]. Suppose $\text{dev}(\gamma)$ and $\text{dev}(\gamma')$ are diffeomorphic by diffeomorphism-germs $\sigma: (\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^2, 0)$ and $\tau: (\mathbf{R}^3, 0) \rightarrow (\mathbf{R}^3, 0)$, and the type of γ' is $(m', m' + s', m' + s' + r')$. In general $\text{dev}(\gamma)$ has singularity always along the original space curve γ , $\{x = 0\}$, and along the tangent line to γ at the origin $\{t = 0, x \neq 0\}$ when $s \geq 2$. Furthermore $\text{dev}(\gamma)$ has the cuspidal edge singularity along $x = 0$, $t \neq 0$, while it has singularity along $\{t = 0, x \neq 0\}$ if and only if $s = 2, r = 1$. On the other hand the curve γ itself is singular if and only if $m \geq 2$. Therefore if the type is not equal to $(1, 3, 4)$, then the diffeomorphism σ preserves $\{x = 0\}$. Then σ and τ have some restrictions: The first component of σ is of form $x\rho(x, t)$, $\rho(0, 0) \neq 0$. The linear term of τ preserves the plane $\{x_1 = 0\}$. Therefore, by the order comparison on t , we see that $s + m = s' + m', r + s + m = r' + s' + m'$. Moreover, restricting the equivalence on γ (and γ'), we see $m = m'$. Hence we have $(m, m + s, m + s + r) = (m', m' + s', m' + s' + r')$. \square

A C^∞ surface (S, γ) with boundary is called of finite type if the boundary γ and the dual-boundary $\hat{\gamma}$ are both of finite type. Note that generic surfaces are of finite type (Lemma 2.18).

From the above argument, in particular we have

Lemma 2.15. *If (S, γ) is of finite type, then the boundary-envelope of (S, γ) is the dual surface $(\hat{\gamma})^\vee$ of the dual-boundary $\hat{\gamma}$. The boundary-envelope is the tangent developable to the dual curve $(\gamma^\vee)^*$ to the dual-boundary γ^\vee . Moreover, if (S, γ) is generic, then there are only cuspidal edge singularities and swallowtail singularities on the boundary-envelope $\pi_1|_{L_2}$.*

Remark 2.16. To investigate the global flat extension problem, we need the global study on singularities of tangent developables. For this subject, see [21].

The following lemma is also a key for the theory:

Lemma 2.17. *Let I be a $(2n + 1)$ -dimensional contact manifold, $\pi: I \rightarrow B$ a Legendre fibration over an $(n + 1)$ -dimensional manifold B , and $k \geq 1$.*

- (1) Define $\Pi: J_{\text{int}}^k(\mathbf{R}, I) \rightarrow J^k(\mathbf{R}, B)$ by $\Pi(j^k \Gamma(t_0)) = j^k(\pi \circ \Gamma)(t_0)$ for any integral curve-germ $\Gamma: (\mathbf{R}, t_0) \rightarrow I$. Then Π is a submersion at $j^k \Gamma(t_0)$ if $\pi \circ \Gamma$ is an immersion at t_0 .
- (2) The set $\Sigma = \{j^k \Gamma(t_0) \in J_{\text{int}}^k(\mathbf{R}, I) \mid \pi \circ \Gamma \text{ is not an immersion at } t_0\}$ is of codimension n in $J_{\text{int}}^k(\mathbf{R}, I)$.

Remark 2.18. By Lemma 2.17 ($n = 2$), we have the following: Let $\Pi_1: J_{\text{int}}^k(\mathbf{R}, I^5) \rightarrow J^k(\mathbf{R}, \mathbf{RP}^3)$ (resp. $\Pi_2: J_{\text{int}}^k(\mathbf{R}, I^5) \rightarrow J^k(\mathbf{R}, \mathbf{RP}^{3*})$) be the mapping induced by the Legendre fibration $\pi_1: I \rightarrow \mathbf{RP}^3$ (resp. $\pi_2: I \rightarrow \mathbf{RP}^{3*}$). Then the set Σ_1 (resp. Σ_2) of jets with singularity after the projection π_1 (resp. π_2) is of codimension 2 in $J_{\text{int}}^k(\mathbf{R}, I^5)$. Moreover $\Pi_1: J_{\text{int}}^k(\mathbf{R}, I^5) \setminus \Sigma_1 \rightarrow J^k(\mathbf{R}, \mathbf{RP}^3) \setminus \Pi_1(\Sigma_1)$ (resp. $\Pi_2: J_{\text{int}}^k(\mathbf{R}, I^5) \setminus \Sigma_2 \rightarrow J^k(\mathbf{R}, \mathbf{RP}^{3*}) \setminus \Pi_2(\Sigma_2)$) is a submersion.

Proof of Lemma 2.17. Take Darboux coordinates $x_1, \dots, x_n, z, p_1, \dots, p_n$ of I around $\Gamma(t_0)$ and x_1, \dots, x_n, z of B around $\pi \circ \Gamma(t_0)$ so that the contact structure is given by $dz - (p_1 dx_1 + \dots + p_n dx_n) = 0$ and π is given by $(x_1, \dots, x_n, z, p_1, \dots, p_n) \mapsto (x_1, \dots, x_n, z)$.

(1) Set $\Gamma(t) = (x_1(t), \dots, x_n(t), z(t), p_1(t), \dots, p_n(t))$ and suppose $\pi \circ \Gamma$ is an immersion at t_0 . Without loss of generality, we suppose $\dot{x}_1(t_0) \neq 0$. Take any deformation $c(t, s) = (X_1(t, s), \dots, X_n(t, s), Z(t, s))$ of $\pi \circ \Gamma(t)$ at $s = 0$. Note that $\dot{z} = p_1 \dot{x}_1 + \dots + p_n \dot{x}_n$. Therefore $p_1(t) = \frac{\dot{z}(t)}{\dot{x}_1(t)} - p_2 \frac{\dot{x}_2(t)}{\dot{x}_1(t)} - \dots - p_n \frac{\dot{x}_n(t)}{\dot{x}_1(t)}$, near $t = t_0$. We set

$$P_1(t, s) := \frac{\dot{Z}(t, s)}{\dot{X}_1(t, s)} - p_2(t) \frac{\dot{X}_2(t, s)}{\dot{X}_1(t, s)} - \dots - p_n(t) \frac{\dot{X}_n(t, s)}{\dot{X}_1(t, s)},$$

$$P_i(t, s) := p_i(t) \quad (i = 2, \dots, n),$$

near $(t, s) = (t_0, 0)$. Here $\dot{Z}(t, s)$ means the derivative by t . Then we get the integral deformation

$$C(t, s) = (X_1(t, s), \dots, X_n(t, s), Z(t, s), P_1(t, s), \dots, P_n(t, s))$$

of $\Gamma(t)$ at $s = 0$, which satisfies $\pi(C(t, s)) = c(t, s)$. This shows that any curve starting at $j^k(\pi \circ \Gamma)(t_0)$ in $J^k(\mathbf{R}, B)$ lifts to a curve starting at $j^k \Gamma(t_0)$ in $J^k_{\text{int}}(\mathbf{R}, I)$. Therefore Π is a submersion at $j^k \Gamma(t_0)$.

To see (2), first remark that $J^k_{\text{int}}(\mathbf{R}, I)$ has local coordinates

$$x_1^{(i)}, \dots, x_n^{(i)}, z, p_1^{(i)}, \dots, p_n^{(i)} \quad (0 \leq i \leq k),$$

because $z^{(i)} \ (2 \leq i \leq k)$ are written by these coordinates from the integrality condition $\dot{z} = p_1 \dot{x}_1 + \dots + p_n \dot{x}_n$. Then Σ is defined exactly by $x'_1 = \dots = x'_n = 0$. \square

Proof of Theorem 2.2. As is mentioned above, we prove Theorem 2.2 using relative version of Theorem 2.4 instead of the ordinary transversality theorem. In fact, we consider three kinds of transversalities: Transversality in $J^k_{\text{int}}(\mathbf{R}^2, I^5)$, that in $J^k_{\text{int}}(\mathbf{R}^2, \mathbf{R}; I^5, I^5)$ and that in $J^k_{\text{int}}(\mathbf{R}, I^5)$. Note that the relative jet space $J^k_{\text{int}}(\mathbf{R}^2, \mathbf{R}; I^5, I^5)$ is fibered over $\mathbf{R} \times I$ with fiber $J^k_{\text{int}}(2, 5)$, the space of jets of integral immersions $(\mathbf{R}^2, 0) \rightarrow (\mathbf{R}^5, 0)$ to a local model \mathbf{R}^5 of the contact space, which is the fibre also for $J^k_{\text{int}}(\mathbf{R}^2, I)$. However we consider the group action on $J^k_{\text{int}}(2, 5)$ for $J^k_{\text{int}}(\mathbf{R}^2, I)$ (resp. $J^k_{\text{int}}(\mathbf{R}^2, \mathbf{R}; I, I)$) by diffeomorphisms on $(\mathbf{R}^2, 0)$ (resp. by relative diffeomorphisms on $(\mathbf{R}^2, \mathbf{R})$) and fiber-preserving contactomorphisms on $(\mathbf{R}^5, 0)$ with a local model $(\mathbf{R}^5, 0) \rightarrow (\mathbf{R}^3)$ of Legendre fibration. We take k sufficiently large. Actually it is enough to take $k \geq 3$ in our case. We use Propositions 2.6 and 2.8. In $J^k_{\text{int}}(\mathbf{R}^2, I)$, we see that the complement to the union of A_ℓ -orbits ($\ell \leq 3$) is of codimension 3 in the jet space of Legendre immersions $J^k_{\text{int}}(\mathbf{R}^2, I)$. Then we have (1) by Theorem 2.4. Moreover, in $J^k_{\text{int}}(\mathbf{R}^2, \mathbf{R}; I, I)$, the complement to the union of B_ℓ - and C_ℓ -orbits ($\ell \leq 3$) is of codimension 2 in $J^k_{\text{int}}(\mathbf{R}^2, \mathbf{R}; I, I)$ along boundary (cf. [1], Theorem 1, Remark 1). Thus, by the relative integral transversality theorem (Remark 2.5), we have (2). In $J^k_{\text{int}}(\mathbf{R}, I)$, by Lemma 2.17 and Remark 2.18, we see that the complement to the jets of integral curves Γ such that $\pi_1 \circ \Gamma$ (resp. $\pi_2 \circ \Gamma$) is Scherbak-generic, is of codimension 2. Therefore, by Theorem 2.4, we have that, for a generic integral curve Γ in I , both $\pi_1|_\Gamma$ and $\pi_2|_\Gamma$ are Scherbak-generic. Therefore by Lemma 2.12, we have (3). \square

3. Euclidean geometry of surface-boundaries

The fundamental construction to observe such characterisations as Theorems 1.8 and 1.9 is as follows:

The unit tangent bundle

$$T_1 \mathbf{R}^3 = \{(x, v) \mid x \in \mathbf{R}^3, v \in T_x \mathbf{R}^3, \|v\| = 1\} \cong \mathbf{R}^3 \times S^2,$$

to the Euclidean three space \mathbf{R}^3 has the contact structure $\{v dx = 0\} \subset T(T_1 \mathbf{R}^3)$. We have analogous double Legendre fibrations as in the projective framework:

$$\begin{array}{ccc} PT^* \mathbf{R}P^3 & \xleftarrow{\varphi} & T_1 \mathbf{R}^3 \\ \pi_1 \searrow & & \searrow \pi_2 \\ \mathbf{R}P^3 \supset \mathbf{R}^3 & & \mathbf{R} \times S^2 \hookrightarrow \mathbf{R}P^3, \end{array}$$

where π_1 is the bundle projection and π_2 is defined by $\pi_2(x, v) = (-x \cdot v, v)$, $\mathbf{R} \times S^2$ being identified with the space of co-oriented affine planes in \mathbf{R}^3 . Note that $T_1 \mathbf{R}^3$ is mapped to $PT^*(\mathbf{R}P^3)$ by $\Phi: (x, v) \mapsto ([1, x], [-x \cdot v, v])$ as a double covering on the image, that the mapping $\Phi: T_1 \mathbf{R}^3 \rightarrow PT^*(\mathbf{R}P^3)$ is a local contactomorphism, and that $\mathbf{R} \times S^2$ is mapped to $\mathbf{R}P^3$ by $(r, v) \mapsto [r, v]$ as a double covering on the image which is $\mathbf{R}P^3 \setminus \{[1, 0, 0, 0]\}$.

Any co-oriented surface with boundary (S, γ) in \mathbf{R}^3 lifts to a Legendre surface with boundary (L, Γ) in $T_1 \mathbf{R}^3$ uniquely. A generic surface in \mathbf{R}^3 induces a generic Legendre surface. The lifted Legendre surface (L, Γ) projects to a front with boundary (boundary-front) in $\mathbf{R} \times S^2$ by π_2 . Actually the “local contact nature” of the double Legendre fibrations is the same, as is noted above, in projective and in Euclidean framework.

Remark 3.1. There exists no invariant metrics on $T_1\mathbf{R}^3$ and on $\mathbf{R} \times S^2$ under the group G of Euclidean motions on \mathbf{R}^3 compatible with the double fibration $\mathbf{R}^3 \leftarrow T_1\mathbf{R}^3 \rightarrow \mathbf{R} \times S^2$. Note that G is not compact. In this sense, there is no dual Euclidean geometry: Duality in the level of Euclidean geometry is not straightforward, compared with projective geometry. As for related result on duality in Euclidean geometry, see [6,5].

Let $S \subset \mathbf{R}^3$ be a co-oriented immersed surface with boundary γ .

The first fundamental form $I:TS \rightarrow \mathbf{R}$ is defined by $I(v) := g_{\text{Eu}}(v, v) = \|v\|^2$. The second fundamental form $II:TS \rightarrow \mathbf{R}$ is defined by $II(v) := -g_{\text{Eu}}(v, \nabla_v \mathbf{n})$, where $\mathbf{n}:S \rightarrow T\mathbf{R}^3$ is the unit normal to S . Then we have $(I, II):TS \rightarrow \mathbf{R}^2$, which determines the surface with boundary essentially.

Set $G = \text{Euclid}(\mathbf{R}^3) \subset \text{GL}(4, \mathbf{R})$, the group of Euclidean motions on \mathbf{R}^3 . We consider Maurer–Cartan form of G ,

$$\omega = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & -\omega_1^2 & -\omega_1^3 \\ \omega^2 & \omega_1^2 & 0 & -\omega_2^3 \\ \omega^3 & \omega_1^3 & \omega_2^3 & 0 \end{pmatrix}.$$

For a surface with boundary, we have the adopted moving frame $\tilde{\gamma} = (\gamma, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3): \mathbf{R} \rightarrow G$ by $\mathbf{e}_1 = \gamma'$, the differentiation by arc-length parameter, \mathbf{e}_2 , the inner normal to γ , and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{n}$, which is different from the Frenet–Serre frame.

The structure equation is given by

$$d(\gamma(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)) = (\gamma(s), \mathbf{e}_1(s), \mathbf{e}_2(s), \mathbf{e}_3(s)) \tilde{\gamma}^* \omega.$$

Thus we have

$$d(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{pmatrix} 0 & -\kappa_1 & -\kappa_2 \\ \kappa_1 & 0 & -\kappa_3 \\ \kappa_2 & \kappa_3 & 0 \end{pmatrix} ds.$$

Namely we have

$$\begin{cases} \mathbf{e}_1' = \kappa_1 \mathbf{e}_2 + \kappa_2 \mathbf{e}_3, \\ \mathbf{e}_2' = -\kappa_1 \mathbf{e}_1 + \kappa_3 \mathbf{e}_3, \\ \mathbf{e}_3' = -\kappa_2 \mathbf{e}_1 - \kappa_3 \mathbf{e}_2. \end{cases}$$

See [16], for instance.

Note that $\kappa_1 = \mathbf{e}_2 \cdot \gamma''$, $\kappa_2 = \mathbf{e}_3 \cdot \gamma''$ and that $\kappa_3 = II(\mathbf{e}_1, \mathbf{e}_2)$.

Suppose (S, γ) is a C^∞ surface with boundary γ . Suppose the boundary $\gamma(t)$ is of finite type at $t = t_0$. Since γ is an immersed curve, the type is written as $(a_1, a_2, a_3) = (1, 1 + s, 1 + s + r)$, for some positive integers r, s .

Then we have

Theorem 3.2. Let (S, γ) be a C^∞ surface with boundary in \mathbf{R}^3 . Suppose γ is of finite $(1, 1 + s, 1 + s + r)$. Then $\gamma(t)$ has an osculating-tangent point at $t = t_0$ if and only if $\kappa_2^{(s-1)}(t_0) = 0$.

Proof. First remark that $\text{rank } A_1(t) = \text{rank } \gamma'(t) = 1$. Then $\text{rank } A_2(t) = \text{rank}(\gamma'(t), \gamma''(t)) = 1$ if and only if $\gamma''(t) (= \mathbf{e}_1'(t))$ is a scalar multiple of $\gamma'(t) (= \mathbf{e}_1)$, and the condition is equivalent to that $\kappa_2(t) = 0$, $\kappa_3(t) = 0$. Similarly we have that $\text{rank } A_i(t) = 1$ ($1 \leq i \leq s$) if and only if $\kappa_1^{(j)}(t) = 0$, $\kappa_2^{(j)}(t) = 0$ ($0 \leq j \leq s - 2$). Then

$$\gamma^{(s+1)}(t) = \mathbf{e}_1^{(s)}(t) = \kappa_1^{(s-1)}(t) \mathbf{e}_2(t) + \kappa_2^{(s-1)}(t) \mathbf{e}_3(t).$$

Moreover we have $\text{rank } A_{s+1}(t) = 2$ if and only if $(\kappa_1^{(s-1)}(t), \kappa_2^{(s-1)}(t)) \neq (0, 0)$. In this case the osculating plane is spanned by $\gamma'(t) = \mathbf{e}_1(t)$, $\gamma^{(s+1)}(t) = \mathbf{e}_1^{(s)}(t)$. Therefore the osculating plane coincides with the tangent plane, which is spanned by $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$, if and only if $\kappa_2^{(s-1)}(t) = 0$. \square

Proof of Theorem 1.8. Generically $\gamma(t)$ is of type $(1, 2, 3)$ or $(1, 2, 4)$. Therefore, applying Theorem 3.2 in the case $s = 1$, the osculating-tangent point is characterised by $\kappa_2 = 0$. \square

The flat extension problem is concerned with osculating-tangent points of the dual boundary $\hat{\gamma}$, not γ . Actually we have

Proposition 3.3. Let (S, γ) be a C^∞ surface with boundary. Suppose $\hat{\gamma}(t)$ is of type $(1, 2, 2 + r)$ at $t = t_0$ for some positive integer r . Then $\hat{\gamma}(t_0)$ is an osculating-tangent point for $(S^\vee, \hat{\gamma})$ if and only if $\kappa_2 = 0$. Therefore, under the above condition, we have that $\gamma(t_0)$ is an osculating-tangent point for (S, γ) if and only if $\hat{\gamma}(t_0)$ is an osculating-tangent point for $(S^\vee, \hat{\gamma})$.

The proof of Proposition 3.3 is given below in the proof of Proposition 3.6.
We show Theorem 1.9 in more general context:

Theorem 3.4. (The characterisation of swallowtail-tangent.) Let (S, γ) be a C^∞ surface with boundary of finite type in \mathbf{R}^3 . Then we have

- (1) A point on the boundary γ is a swallowtail-tangent point with (I) $\kappa_2 \neq 0$ if and only if the conditions (II), (III) of Theorem 1.9 hold.
- (2) A point on the boundary γ is a swallowtail-tangent point with $\kappa_2 = 0$ if and only if (I)' $\kappa_1 \neq 0$, $\kappa_3 \neq 0$, (II)' $\kappa'_2 = \frac{1}{2}\kappa_1\kappa_3$, and (III)' $\kappa''_3 \neq \frac{4}{3}\kappa_3\kappa'_1$.

Remark 3.5.

- (1) A swallowtail-tangent point with $\kappa_2 = 0$ does not appear generically.
- (2) The criteria of Theorem 3.4 has the similarity in the form to the general criterion of swallowtail found in [17,22].

Proof of Theorem 3.4. The dual-boundary $\hat{\gamma}$ is given by $(\mathbf{n}, -\gamma \cdot \mathbf{n}) : (\mathbf{R}, 0) \rightarrow S^2 \times \mathbf{R}$. Since $S^2 \times \mathbf{R}$ is mapped to $\mathbf{RP}^{3*} \setminus \{[1, 0, 0, 0]\}$ as a double covering, $\hat{\gamma}$ is regarded as a curve in \mathbf{RP}^{3*} . To see the type of $\hat{\gamma}$ we examine the $4 \times (r+1)$ matrix

$$\tilde{A}_r(t) = \begin{pmatrix} \mathbf{n}(t) & \mathbf{n}'(t) & \mathbf{n}''(t) & \cdots & \mathbf{n}^{(r)}(t) \\ -\gamma \cdot \mathbf{n}(t) & (-\gamma \cdot \mathbf{n})'(t) & (-\gamma \cdot \mathbf{n})''(t) & \cdots & (-\gamma \cdot \mathbf{n})^{(r)}(t) \end{pmatrix}$$

($r = 1, 2, \dots$). In terms of homogeneous coordinates, the curve $\hat{\gamma}(t)$ is of type (a_1, a_2, a_3) at $t = t_1$ if and only if,

$$\begin{aligned} \min\{r \mid \text{rank } \tilde{A}_r(t_0) = 2\} &= a_1, & \min\{r \mid \text{rank } \tilde{A}_r(t_0) = 3\} &= a_2, \\ \min\{r \mid \text{rank } \tilde{A}_r(t_0) = 4\} &= a_3. \end{aligned}$$

In fact $\text{rank } \tilde{A}_r(t) = \text{rank } A_r(t) + 1$, for the matrix $A_r(t)$ introduced in Section 2.

As is mentioned, the boundary-envelope of (S, γ) , namely, the dual surface to the dual-boundary $\hat{\gamma}$ has the cuspidal edge along the dual curve $\hat{\gamma}^*$ of $\hat{\gamma}$, where $\hat{\gamma}(t)$ is of type $(1, 2, 3)$. The curve $\hat{\gamma}$ in \mathbf{RP}^{3*} is of type $(1, 2, 3)$ at $t = t_0$ if and only if $\det A_3(t_0) \neq 0$. In fact the condition is equivalent to that

$$\det(\hat{\gamma}'(t_0), \hat{\gamma}''(t_0), \hat{\gamma}'''(t_0)) \neq 0.$$

Similarly, the boundary-envelope of (S, γ) is diffeomorphic to the swallowtail at the point $\hat{\gamma}^*(t_1)$ in $\mathbf{R}^3 \subset \mathbf{RP}^3$ if and only if $\hat{\gamma}(t)$ is of type $(1, 2, 4)$ at $t = t_1$. The condition is equivalent to that

$$\text{rank } \tilde{A}_1 = 2, \quad \text{rank } \tilde{A}_2 = 3, \quad \text{rank } \tilde{A}_3 = 3, \quad \text{rank } \tilde{A}_4 = 4,$$

at $t = t_1$. Then, by the straightforward calculation, using the structure equation explained above, we have the criteria in Theorem 3.4. In fact, from $\gamma' = \mathbf{e}_1$, $\mathbf{n} = \mathbf{e}_3$, $\gamma'' = \kappa_1\mathbf{e}_2 + \kappa_2\mathbf{e}_3$, we have

$$\begin{aligned} \gamma''' &= -(\kappa_1^2 + \kappa_2^2)\mathbf{e}_1 + (\kappa'_1 - \kappa_2\kappa_3)\mathbf{e}_2 + (\kappa'_2 + \kappa_1\kappa_3)\mathbf{e}_3, \\ \gamma'''' &= (-3\kappa_1\kappa'_1 - 3\kappa_2\kappa'_2)\mathbf{e}_1 + (-\kappa_1^3 - \kappa_1\kappa_2^2 - \kappa_1\kappa_3^2 + 2\kappa'_1\kappa_3 + \kappa_1\kappa'_3 + \kappa'_2)\mathbf{e}_3. \end{aligned}$$

Moreover we have $\gamma'' \cdot \mathbf{n}' = \kappa_2$, $\gamma'' \cdot \mathbf{n}'' = -\kappa_1\kappa_3$. Thus we have

$$\begin{aligned} (\gamma \cdot \mathbf{n})' - \gamma \cdot \mathbf{n}' &= 0, \\ (\gamma \cdot \mathbf{n})'' - \gamma \cdot \mathbf{n}'' &= -\kappa_2, \\ (\gamma \cdot \mathbf{n})''' - \gamma \cdot \mathbf{n}''' &= -2\kappa'_2 + \kappa_1\kappa_3, \\ (\gamma \cdot \mathbf{n})'''' - \gamma \cdot \mathbf{n}'''' &= \kappa_1^2\kappa_2 + \kappa_2^3 + \kappa_2\kappa_3^2 + 2\kappa_3\kappa'_1 + 3\kappa_1\kappa'_3 - 3\kappa''_3. \end{aligned}$$

Then the condition $\text{rank } \tilde{A}_1(t_1) = 2$ is equivalent to that $\kappa_2 \neq 0, \kappa_3 \neq 0$ at $t = t_1$. The condition $\text{rank } \tilde{A}_2(t_1) = 3$ is equivalent to that $\kappa_2 \neq 0$, or $\kappa_2 = 0, \kappa_3 \neq 0, \kappa_2\kappa_3 - \kappa'_2 \neq 0$ at $t = t_1$.

Let us see the condition $\text{rank } \tilde{A}_3(t_1) = 3$, namely that $\det(\tilde{A}_3(t_1)) = 0$. We set $D = \det(\tilde{A}_3(t_1))$. Then we have, after simplifying the determinant and taking the transpose of \tilde{A}_3 ,

$$D = \begin{vmatrix} \mathbf{e}_3 & 0 \\ -\kappa_2\mathbf{e}_1 - \kappa_3\mathbf{e}_2 & 0 \\ (\kappa_1\kappa_3 - \kappa'_2)\mathbf{e}_1 + (-\kappa_1\kappa_2 - \kappa'_3)\mathbf{e}_2 & \kappa_2 \\ A\mathbf{e}_1 + B\mathbf{e}_2 & 2\kappa'_2 - \kappa_1\kappa_3 \end{vmatrix},$$

where we set $\mathbf{n}''' = A\mathbf{e}_1 + B\mathbf{e}_2 + C\mathbf{e}_3$,

$$\begin{aligned} A &= \kappa_3\kappa_1' + 2\kappa_1\kappa_3' - \kappa_2'' + \kappa_1^2\kappa_2 + \kappa_2^3 + \kappa_2\kappa_3^2, \\ B &= -\kappa_2\kappa_1' - 2\kappa_1\kappa_2' - \kappa_3'' + \kappa_1^2\kappa_3 + \kappa_2^2\kappa_3 + \kappa_3^2, \\ C &= -3\kappa_2\kappa_2' - 3\kappa_3\kappa_3'. \end{aligned}$$

Then we see that D is equal to the left-hand side of the condition (II) of Theorem 1.9.

To see the condition $\text{rank } \tilde{A}_4(t_1) = 4$ we calculate the sub-determinant E obtained by deleting the fourth column from $\tilde{A}_4(t_1)$. The condition is equivalent to $E \neq 0$. The sub-determinant E is given by

$$E = - \begin{vmatrix} \kappa_2 & \kappa_2' - \kappa_1\kappa_3 & A' - B\kappa_1 \\ \kappa_3 & \kappa_3' + \kappa_1\kappa_2 & B' - A\kappa_1 \\ 0 & \kappa_2 & \kappa_1^2\kappa_2 + \kappa_2^3 + \kappa_2\kappa_3^2 + 2\kappa_3\kappa_1' + 3\kappa_1\kappa_3' - 3\kappa_3'' \end{vmatrix},$$

and it is equal to, up to sign, the left-hand side of (III). Thus we have (1).

To see (2), suppose the non-generic condition $\kappa_2 = 0$. Then we have $\kappa_3 \neq 0$, $\kappa_1\kappa_3 - \kappa_2' \neq 0$. From the condition $D = 0$ we have

$$\kappa_1^2\kappa_3^2 - 3\kappa_1\kappa_2\kappa_2' + 2(\kappa_2')^2 = 0. \quad (*)$$

From the condition $E \neq 0$, we have

$$2\kappa_1\kappa_3^2\kappa_1' - 3(\kappa_1\kappa_3 - \kappa_2')\kappa_3'' \neq 0. \quad (**)$$

Since Eq. (*) has solutions $\kappa_2' = \kappa_1\kappa_3$, $\frac{1}{2}\kappa_1\kappa_3$, we have $\kappa_2' = \frac{1}{2}\kappa_1\kappa_3$ and $\kappa_1 \neq 0$. Then the condition (**) is equivalent to that $\kappa_3'' \neq \frac{4}{3}\kappa_3\kappa_1'$. \square

Proof of Remark 1.10. At an osculating point on γ , the second fundamental form II of S satisfies $II(\mathbf{e}_1, \mathbf{e}_1) = -\mathbf{e}_3' \cdot \mathbf{e}_1 = \kappa_2 = 0$ and $II(\mathbf{e}_1, \mathbf{e}_2) = \kappa_3$. Therefore $\det(II) = -\kappa_3^2 \leq 0$. Moreover $\det(II) < 0$ if and only if $\kappa_3 \neq 0$. \square

The envelope-swallowtail point for (S, γ) corresponds to the osculating plane to $\hat{\gamma}$ at a point $t = t_1$ of type $(1, 2, 4)$ in \mathbf{RP}^{3*} . Then d is the distance between $\gamma(t_1)$ and $\hat{\gamma}^*(t_1)$. Actually the formula in Proposition 1.11 gives $\text{dist}(\gamma(t), \hat{\gamma}^*(t))$:

Proposition 3.6. Let (S, γ) be a C^∞ surface with boundary. Suppose the dual-boundary $\hat{\gamma}(t)$ is of type $(1, 2, 2+r)$ at $t = t_1$ for some positive integer r . Then the distance $d = \text{dist}(\gamma(t_1), \hat{\gamma}^*(t_1))$ is given by

$$d = \left| \frac{\kappa_2 \sqrt{\kappa_2^2 + \kappa_3^2}}{\kappa_2(\kappa_3' + \kappa_1\kappa_2) + \kappa_3(-\kappa_2' + \kappa_1\kappa_3)} \right|$$

at $t = t_1$.

Proposition 1.11 follows from Proposition 3.6.

Proof of Propositions 3.3 and 3.6. Let $\gamma(t)$ be a point on the boundary γ in \mathbf{RP}^3 . Set $\hat{\gamma}(t) = (-\gamma(t) \cdot \mathbf{n}(t), \mathbf{n}(t))$, where t is the arc-length parameter. Since $\hat{\gamma}$ is of type $(1, 2, 2+r)$, $\hat{\gamma}'$, $\hat{\gamma}''$ are linearly independent. Then the point $\hat{\gamma}^*(t) = [1, x] = [1, x_1, x_2, x_3] \in \mathbf{R}^3 \subset \mathbf{RP}^3$ is obtained by solving the system of equations

$$\begin{cases} \mathbf{x} \cdot \mathbf{n} - \gamma(t) \cdot \mathbf{n}(t) = 0, \\ \mathbf{x} \cdot \mathbf{n}' - (\gamma(t) \cdot \mathbf{n}(t))' = 0, \\ \mathbf{x} \cdot \mathbf{n}'' - (\gamma(t) \cdot \mathbf{n}(t))'' = 0. \end{cases}$$

We set $\Delta = \begin{vmatrix} n_1 & n_2 & n_3 \\ n_1' & n_2' & n_3' \\ n_1'' & n_2'' & n_3'' \end{vmatrix}$, where we set $\mathbf{n}(t) = (n_1(t), n_2(t), n_3(t))$. Note that, under the assumption, the Gauss mapping $\mathbf{n}(t)$ restricted at γ is immersive and therefore $\Delta \neq 0$. Then, by Cramér's formula, we have

$$\begin{aligned} x_1 &= \frac{1}{\Delta} \begin{vmatrix} \gamma \cdot \mathbf{n} & n_2 & n_3 \\ (\gamma \cdot \mathbf{n})' & n_2' & n_3' \\ (\gamma \cdot \mathbf{n})'' & n_2'' & n_3'' \end{vmatrix}, & x_2 &= \frac{1}{\Delta} \begin{vmatrix} n_1 & \gamma \cdot \mathbf{n} & n_3 \\ n_1' & (\gamma \cdot \mathbf{n})' & n_3' \\ n_1'' & (\gamma \cdot \mathbf{n})'' & n_3'' \end{vmatrix}, \\ x_3 &= \frac{1}{\Delta} \begin{vmatrix} n_1 & n_2 & \gamma \cdot \mathbf{n} \\ n_1' & n_2' & (\gamma \cdot \mathbf{n})' \\ n_1'' & n_2'' & (\gamma \cdot \mathbf{n})'' \end{vmatrix}. \end{aligned}$$

Since

$$(\gamma \cdot \mathbf{n})' = \gamma' \cdot \mathbf{n} + \gamma \cdot \mathbf{n}' = \gamma \cdot \mathbf{n}', \quad (\gamma \cdot \mathbf{n})'' = \gamma' \cdot \mathbf{n}' + \gamma \cdot \mathbf{n}'', \quad \gamma' \cdot \mathbf{n}' = -\kappa_2,$$

we have

$$x_1 = \gamma_1 + \frac{1}{\Delta} \begin{vmatrix} 0 & n_2 & n_3 \\ 0 & n_2' & n_3' \\ \gamma' \cdot \mathbf{n}' & n_2'' & n_3'' \end{vmatrix} = \gamma_1 - \frac{\kappa_2}{\Delta} \begin{vmatrix} n_2 & n_3 \\ n_2' & n_3' \end{vmatrix}.$$

Similarly we have

$$x_2 = \gamma_2 + \frac{\kappa_2}{\Delta} \begin{vmatrix} n_1 & n_3 \\ n_1' & n_3' \end{vmatrix}, \quad x_3 = \gamma_3 - \frac{\kappa_2}{\Delta} \begin{vmatrix} n_1 & n_2 \\ n_1' & n_2' \end{vmatrix}.$$

The distance d between a point $\gamma(t)$ on the boundary and the point $\widehat{\gamma}^*(t)$ on the boundary-envelopes is calculated by

$$\begin{aligned} d^2 &= (x_1 - \gamma_1)^2 + (x_2 - \gamma_2)^2 + (x_3 - \gamma_3)^2 \\ &= \frac{\kappa_2^2}{\Delta^2} \left(\begin{vmatrix} n_2 & n_3 \\ n_2' & n_3' \end{vmatrix}^2 + \begin{vmatrix} n_1 & n_3 \\ n_1' & n_3' \end{vmatrix}^2 + \begin{vmatrix} n_1 & n_2 \\ n_1' & n_2' \end{vmatrix}^2 \right). \end{aligned}$$

Now, from the structure equation, we have

$$\begin{aligned} \Delta &= |\mathbf{e}_3, \mathbf{e}_3', \mathbf{e}_3''| \\ &= |\mathbf{e}_3, -\kappa_2 \mathbf{e}_1 - \kappa_3 \mathbf{e}_2, (-\kappa_2' + \kappa_1 \kappa_3) \mathbf{e}_1 - (\kappa_3' + \kappa_1 \kappa_2) \mathbf{e}_2 - (\kappa_2^2 + \kappa_3^2) \mathbf{e}_3| \\ &= \kappa_2 (\kappa_3' + \kappa_1 \kappa_2) + \kappa_3 (-\kappa_2' + \kappa_1 \kappa_3). \end{aligned}$$

On the other hand, for the exterior product,

$$\begin{aligned} \mathbf{n} \times \mathbf{n}' &= *(\mathbf{n} \wedge \mathbf{n}') = \mathbf{e}_3 \times (-\kappa_2 \mathbf{e}_1 - \kappa_3 \mathbf{e}_2) = \kappa_3 \mathbf{e}_1 - \kappa_2 \mathbf{e}_2, \\ |\mathbf{n} \times \mathbf{n}'|^2 &= |\kappa_3 \mathbf{e}_1 - \kappa_2 \mathbf{e}_2|^2 = \kappa_2^2 + \kappa_3^2. \end{aligned}$$

Therefore

$$d^2 = \frac{\kappa_2^2 (\kappa_2^2 + \kappa_3^2)}{[\kappa_2 (\kappa_3' + \kappa_1 \kappa_2) + \kappa_3 (-\kappa_2' + \kappa_1 \kappa_3)]^2}.$$

Hence we have the formula of Proposition 3.6 and therefore Proposition 1.11. Moreover, we see $\widehat{\gamma}^*(t_1)$ coincides with $\gamma(t_1)$ if and only if $\kappa_2^2 (\kappa_2^2 + \kappa_3^2) = 0$, which is equivalent to that $\kappa_2 = 0$ at $t = t_1$. Thus we have Proposition 3.3. \square

To show Theorem 1.1, we show first

Lemma 3.7. *Let (S, γ) be a C^∞ surface with boundary, \widetilde{S} a flat C^1 extension of S . Suppose the restriction $g|_\gamma$ of the Gauss mapping of S restricted on γ is an immersion. Then for any $p \in \gamma$, there is an open neighbourhood U of p in $\widetilde{S} \setminus \text{Int } S$ such that the Legendre lifting of U projects by π_2 to $\widehat{\gamma}$.*

Proposition 3.8. *Set $S' = \widetilde{S} \setminus \text{Int } S$. Note that S' is a C^∞ surface with boundary γ . Consider the Legendre liftings (L, Γ) of (S, γ) and (L', Γ') of S' in the incident manifold I^5 with respect to the projection π_1 . Because \widetilde{S} is a C^1 surface, we see $\widetilde{L} = L \cup L'$ is a C^0 surface in I . From the assumption that $g|_\gamma$ is immersive, we see the S^2 component of $\pi_2|_\Gamma : \Gamma \rightarrow S^2 \times \mathbf{R}$ is immersive. Consider the Gauss mapping g' of S' and its restriction $g'|_\gamma$. Then $g'|_\gamma$ is immersive if and only if the S^2 -component of $\pi_2|_\Gamma$ is immersive. Since S' is flat, g' is of rank < 2 . Hence g' is of rank one along γ . Therefore $\pi_2|_{L'}$ is of rank one along Γ . Moreover the kernel field of $\pi_2|_{L'}$ is transverse to Γ on Γ . Then L' projects to $\widehat{\gamma}$ near Γ . \square*

Proof of Theorem 1.1. Suppose (S, γ) is generic. Then the dual-boundary $\widehat{\gamma}$ is of type $(1, 2, 3)$ or $(1, 2, 4)$ (Theorem 2.2). Suppose $p \in \gamma$ is not an osculating-tangent point. The boundary-envelope E is non-singular near p (Theorem 3.2). Then, actually, the pair (S, E) is of type B_2 and (S, γ) has the C^1 flat extension by E . To show the uniqueness of local flat extensions, suppose (S, γ) has a local C^1 flat extension \widetilde{S} . Then by Lemma 3.7 the Legendre lifting L' of $\widetilde{S} \setminus \text{Int } S$ projects to $\widehat{\gamma}$ locally at each point of Γ . Therefore L' is contained in the projective conormal bundle of $\pi_2|_\Gamma$. Hence, by projecting by π_1 , we see that $\widetilde{S} \setminus \text{Int } S$ is locally contained in the boundary-envelope $\pi_1(L_2)$. Thus we have the local uniqueness of the flat extension. \square

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